5.1 Eigenvalues and Eigenvectors

- Diagonalization
- Eigenvalues and Eigenvectors
- Characteristic Polynomial
- Properties

Diagonalization

Definition

A linear operator T on a finite-dimensional vector space V is diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is diagonalizable if L_A is diagonalizable.

Eigenvalues and Eigenvectors

Definition

Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is an eigenvector of T if there exists a scalar eigenvalue λ corresponding to the eigenvector v such that $T(v) = \lambda v$.

Let $A \in M_{n\times n}(F)$. A nonzero vector $v \in F^n$ is an eigenvector of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar eigenvalue λ of A corresponding to the eigenvector v.

Eigenvalues and Eigenvectors: Example

Example

Let $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Examine the images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A.

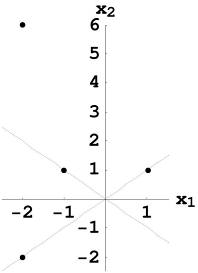
Solution

$$A\mathbf{u} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\mathbf{u}$$

$\mathbf{u}$ is called an *eigenvector* of A since $A\mathbf{u}$ is a multiple of $\mathbf{u}$.

$$A\mathbf{v} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \neq \lambda\mathbf{v}$$

$\mathbf{v}$ is not an eigenvector of A since $A\mathbf{v}$ is not a multiple of $\mathbf{v}$.



$A\mathbf{u} = -2\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$

Eigenvalues and Eigenvectors: Example

Example

Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

Solution: Scalar 4 is an eigenvalue of A if and only if $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution.

$$A\mathbf{x} - 4\mathbf{x} = \mathbf{0}$$

$$A\mathbf{x} - 4\left(\_\_\_\_\right)\mathbf{x} = \mathbf{0}$$

$$(A - 4I)\,\mathbf{x} = \mathbf{0}.$$

To solve $(A - 4I)\,\mathbf{x} = \mathbf{0}$, we need to find $A - 4I$ first:

$$A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}$$

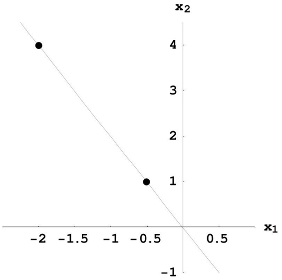
Eigenvalues and Eigenvectors: Example

Now solve $(A - 4I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \qquad \mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Each vector of the form $x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$.



Eigenspace for $\lambda = 4$

The set of all solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of A corresponding to λ.

Diagonalization

Theorem (5.1)

A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T. If T is diagonalizable, $\beta = \{v_1, \cdots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.

Diagonalization

To diagonalize a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.

Characteristic Polynomial

Theorem (5.2)

Let $A \in M_{n\times n}(F)$*. Then a scalar* λ *is an eigenvalue of* A *if and only if* $\det(A - \lambda I_n) = 0$.

Characteristic Polynomial

Definition

Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the characteristic polynomial of A.

Characteristic Polynomial

Definition

Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the characteristic polynomial $f(t)$ of T to be the characteristic polynomial of $A = [T]_\beta$:
$f(t) = \det(A - tI_n)$.

Properties

Theorem (5.3)

Let $A \in M_{n\times n}(F)$.

(a) *The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.*

(b) *A has at most n distinct eigenvalues.*

Properties

Theorem (5.4)

Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.

5.2 Diagonalizability

- Diagonalizability
- Multiplicity
- Direct Sums

Diagonalizability

Theorem (5.5)

Let T be a linear operator on a vector space V, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of T. If $v_1, \cdots, v_k$ are the corresponding eigenvectors, then $\{v_1, \cdots, v_k\}$ is linearly independent.

Corollary

Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

Diagonalizability (cont.)

Definition

A polynomial $f(t)$ in $P(F)$ *splits over* F if there are scalars c, a_1, $\cdots$, a_n in F such that $f(t) = c(t - a_1)(t - a_2)\cdots(t - a_n)$.

Theorem (5.6)

The characteristic polynomial of any diagonalizable operator splits.

Multiplicity

Definition

Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The (algebraic) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$.

Definition

Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define $E_\lambda = \{x \in V : T(x) = \lambda x\} = N(T - I_V)$. The set E_λ is the eigenspace of T corresponding to the eigenvalue λ. The eigenspace of a square matrix A is the eigenspace of L_A.

Multiplicity (cont.)

Theorem (5.7)

Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \leq \dim(E_\lambda) \leq m$.

Diagonalizability

Lemma

Let T be a linear operator, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of T. For $i = 1, \cdots, k$, let $v_i \in E_{\lambda_i}$. If

$$v_1 + v_2 + \cdots + v_k = 0,$$

then $v_i = 0$ for all i.

Theorem (5.8)

Let T be a linear operator on a vector space V, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of T. For $i = 1, \cdots, k$, let S_i be a finite linearly independent subset of the eigenspace E_{λ_i}. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of V.

Diagonalizability

Theorem (5.9)

Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T. Then

(a) *T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for all i.*

(b) *If T is diagonalizable and β_i is an ordered basis for E_{λ_i}, for each i, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.*

Diagonalizability (cont.)

Test for Diagonalization

Let T be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if both of the following conditions hold.

- The characteristic polynomial of T splits.
- The multiplicity of each eigenvalue λ equals $n - \mathrm{rank}(T - \lambda I)$.

Direct Sums

Definition

The sum of the subspaces $W_1, \cdots, W_k$ of a vector space is the set

$$\sum_{i=1}^{k} W_i = \{v_1 + \cdots + v_k : v_i \in W_i \text{ for } 1 \leq i \leq k\}.$$

Definition

A vector space V is the direct sum of subspaces $W_1, \cdots, W_k$, denoted $V = W_1 \oplus \cdots \oplus W_k$, if

$$V = \sum_{i=1}^{k} W_i \text{ and } W_j \cap \sum_{i \neq j} W_i = \{0\} \text{ for each } j, 1 \leq j \leq k.$$

Direct Sums (cont.)

Theorem (5.10)

Let $W_1, \cdots, W_k$ be subspaces of finite-dimensional vector space V. The following are equivalent:

(a) $V = W_1 \oplus \cdots \oplus W_k$.

(b) $V = \sum_{i=1}^{k} W_i$ *and for any* $v_1, \cdots, v_k$ *s.t.* $v_i \in W_i$ $(1 \leq i \leq k)$*, if* $v_1 + \cdots + v_k = 0$*, then* $v_i = 0$ *for all* i.

(c) *Each* $v \in V$ *can be uniquely written as* $v = v_1 + \cdots + v_k$*, where* $v_i \in W_i$.

(d) *If* γ_i *is an ordered basis for* W_i $(1 \leq i \leq k)$*, then* $\gamma_1 \cup \cdots \cup \gamma_k$ *is an ordered basis for* V.

(e) *For each* $i = 1, \cdots, k$ *there exists an ordered basis* γ_i *for* W_i *such that* $\gamma_1 \cup \cdots \cup \gamma_k$ *is an ordered basis for* V.

Direct Sums (cont.)

Theorem (5.11)

A linear operator T on finite-dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigenspaces of T.

5.3 Matrix Limites and Markov Chains

- Matrix Limits
- Existence of Limits

Matrix Limits

Definition

Let L, A_1, A_2, $\cdots$ be $n \times p$ matrices with complex entries. The sequence A_1, A_2, $\cdots$ is said to *converge to the limit* L if $\lim_{m\to\infty}(A_m)_{ij} = L_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq p$. If L is the limit of the sequence, we write $\lim_{m\to\infty} A_m = L$.

Theorem (5.12)

Let A_1, A_2, $\cdots$ be a sequence of $n \times p$ matrices with complex entries that converges to L. Then for any $P \in M_{r\times n}(C)$ and $Q \in M_{p\times s}(C)$,

$$\lim_{m\to\infty} PA_m = PL \text{ and } \lim_{m\to\infty} A_mQ = LQ.$$

Matrix Limits (cont.)

Corollary

Let $A \in M_{n\times n}(C)$ be such that $\lim_{m\to\infty} A^m = L$. Then for any invertible $Q \in M_{n\times n}(C)$,

$$\lim_{m\to\infty} (QAQ^{-1})^m = QLQ^{-1}.$$

Existence of Limits

Consider the set consisting of the complex number 1 and the interior of the unit disk: $S = \{\lambda \in \mathbb{C} : |\lambda| < 1 \text{ or } \lambda = 1\}$.

Theorem (5.13)

Let A be a square matrix with complex entries. Then $\lim_{m\to\infty} A^m$ exists if and only if both of the following hold:

(a) *Every eigenvalue of A is contained in S.*

(b) *If 1 is an eigenvalue of A, then the dimension of the eigenspace corresponding to 1 equals the multiplicity of 1 as an eigenvalue of A.*

Existence of Limits (cont.)

Theorem (5.14)

Let $A \in M_{n\times n}(C)$. $\lim_{m\to\infty} A^m$ exists if

(a) *Every eigenvalue of A is contained in S.*

(b) *A is diagonalizable.*